Morphological Data Analysis

L. Najman

Joint work with Aditya Challa,
Sravan Danda and B.S. Daya Sagar

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Mathematical morphology

- is a theory and technique for the analysis and processing of geometrical structures,
  - based on set theory, lattice theory, topology, and random functions.
- most commonly applied to digital images,
  - but it can be employed as well on graphs, surface meshes, solids, and many other spatial structures.
- Basic morphological operators are erosion, dilation, opening and closing

Plan

- MorphMedian and semi-supervised clustering
  - The watershed as a classifier
- Some links with optimization framework
  - The Power Watershed framework
    - Random walker, spectral clustering

Part I: Morphological Median and the watershed

Morphological Median

Interpolation of shapes

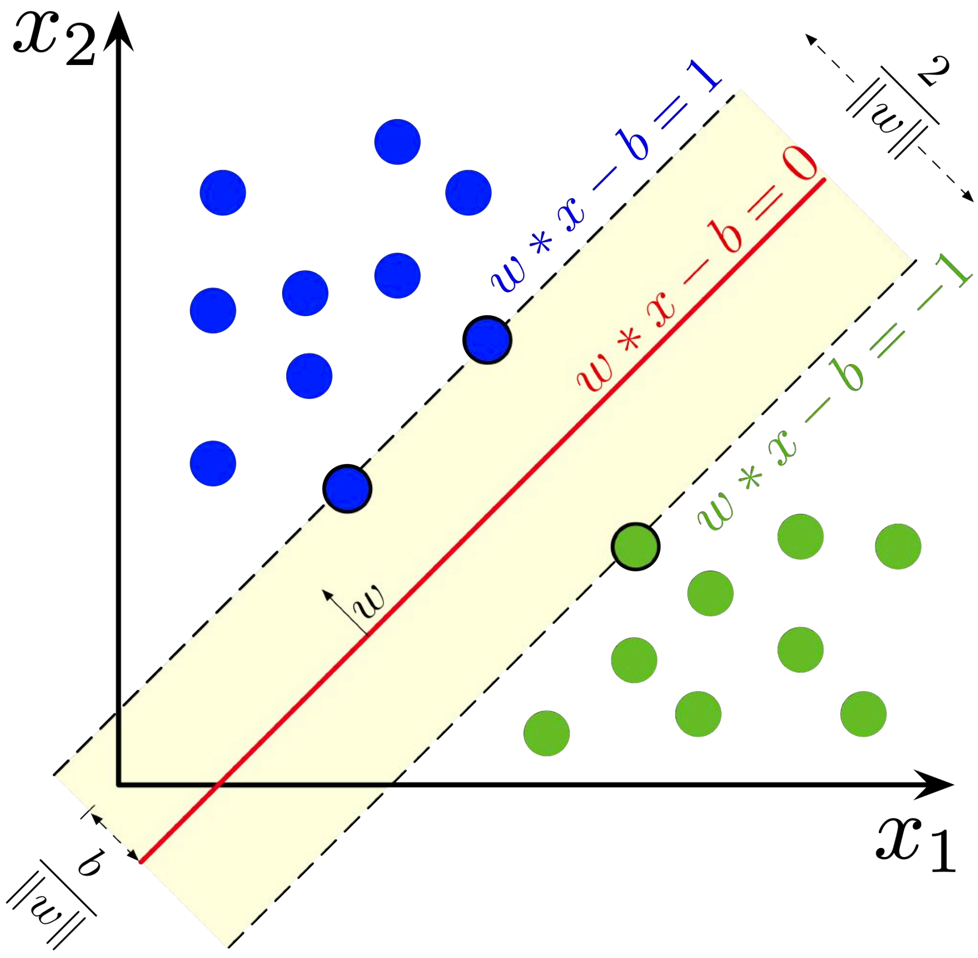
$$M(X,Y) = \bigcup_{\lambda \geq 0} \{(X \oplus \lambda B) \cap (Y \ominus \lambda B)\}$$

Rewriting Morphological Median

$$d(X, Z) = \inf\{\lambda \mid Z \subseteq X \oplus \lambda B\}$$

$$M(X, Y) = \{x | d(X, x) \leq d(Y^c, x)\} = IZ(X \mid Y^c)$$

Linear SVM: maximum margin



"Minimize $\|\vec{w}\|$ subject to $y_i(\vec{w} \cdot \vec{x}_i - b) \geq 1$ for $i = 1, \ldots, n$."

The Maximum Margin Partition

$$
\begin{aligned}
V &= \text{Some set of points} \\
\rho(x, y) &:= \text{Dissimilarity between } x \text{ and } y \\
\rho(X, Y) &:= \inf_{x \in X, y \in Y} \rho(x, y) \\
X_0 = \text{Label 0 set} & \qquad X_1 = \text{Label 1 set} \\
\text{(Require)} V = M_0 \cup M_1 & \qquad X_0 \subset M_0, X_1 \subset M_1
\end{aligned}
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Observe:

$$\mathit{Margin}(X_0, \text{boundary}) = \rho(X_0, M_1)$$

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Observe:

$$\begin{aligned} Margin(X_0, \text{boundary}) &= \rho(X_0, M_1) \\ Margin(X_1, \text{boundary}) &= \rho(X_1, M_0) \end{aligned}$$

The Maximum Margin Partition

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\end{aligned}
$$

$$
Margin = \inf\{\rho(X_0, M_1), \rho(X_1, M_0)\}
$$

The Maximum Margin Partition

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\end{aligned}
$$

Result (Maximum Margin Partition)

Given the definitions as above, a partition $V = M_0 \cup M_1$ is called the maximum partition if it is

$$\arg\max_{M_0, M_1} \inf \{\rho(X_0, M_1), \rho(X_1, M_0)\}$$

The Maximum Margin Partition

$$\arg\max_{M} \hat{\rho}(X_0, X_1, M) = \arg\max_{M} \left\{\inf \left\{\rho(X_0, \overline{M}), \rho(X_1, M)\right\}\right\}$$

$$\rho(X, Y) = \inf_{x \in X, y \in Y} \rho(x, y).$$

MorphMedian

Recall:

$$m(X, Y) = \{x | d(X, x) \leq d(Y^c, x)\}$$

MorphMedian

Result

Given (V, ρ)*, and* X_0, X_1*, every maximum margin partition is* MORPHMEDIAN *and vice versa.*

MorphMedian

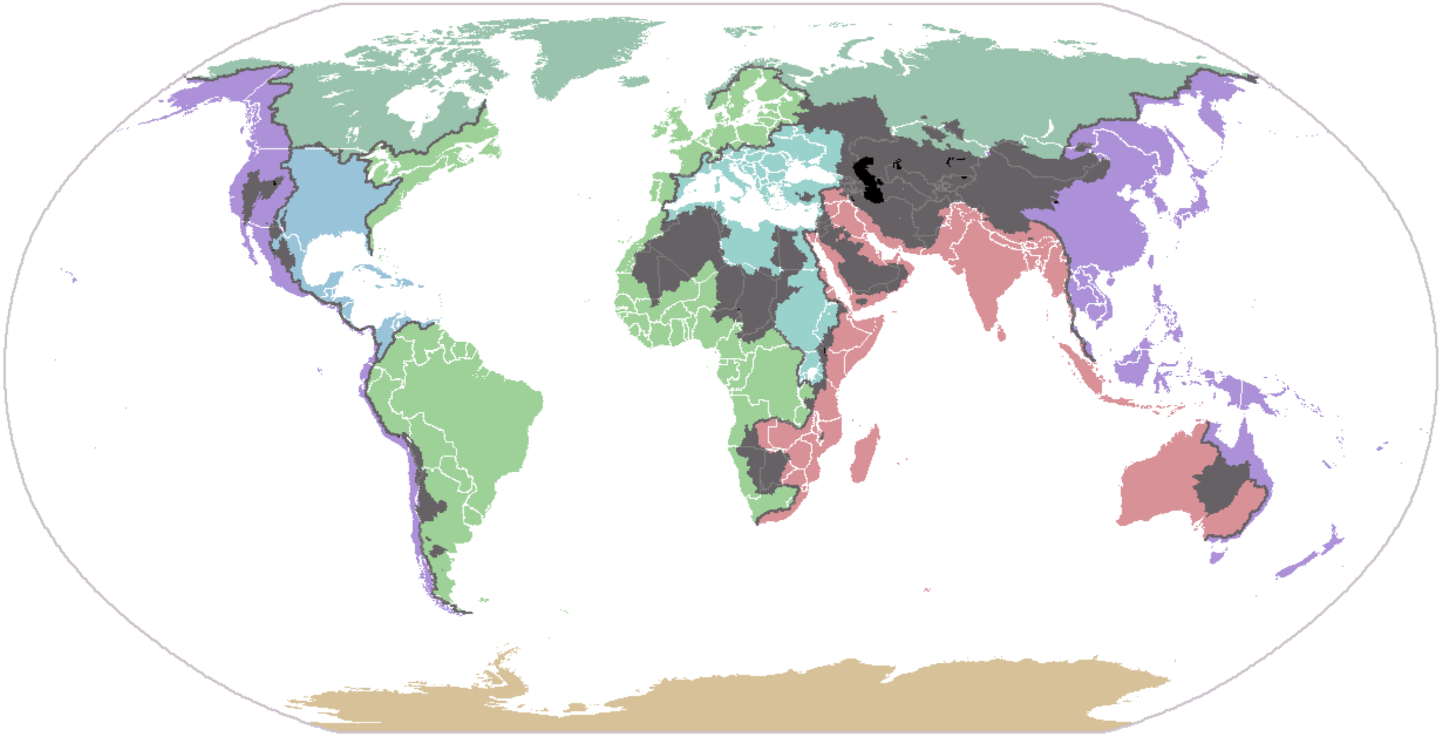
$V =$ Set of points
$\rho :=$ Dissimilarity Measure

Result (morphMedian)

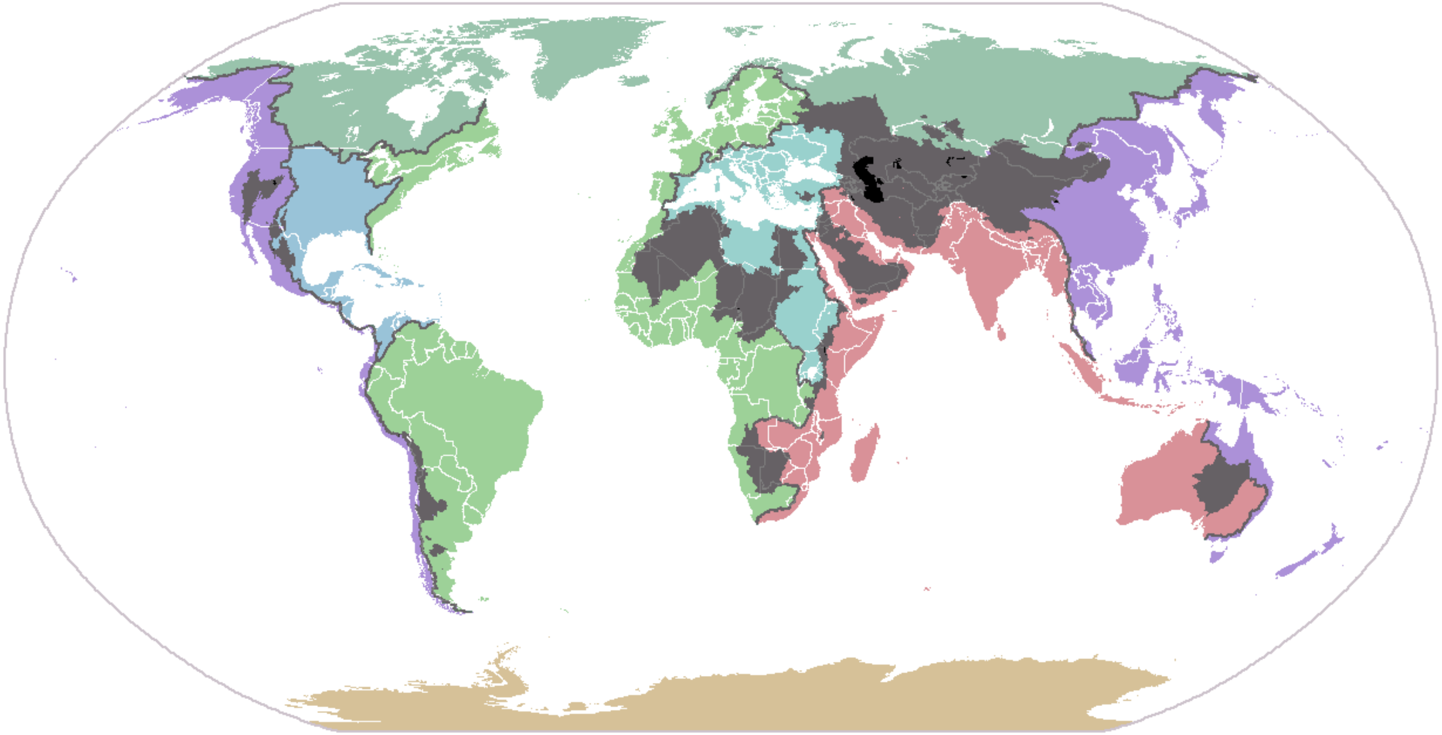
Let (V, ρ) be defined as above. X_0, X_1 denote the labelled sets. Define the morphMedian *partition as any partition which satisfies*

1. $x \in X_0$ *if* $\rho(X_0, x) < \rho(X_1, x)$
2. $x \in X_1$ *if* $\rho(X_1, x) < \rho(X_0, x)$

Watersheds



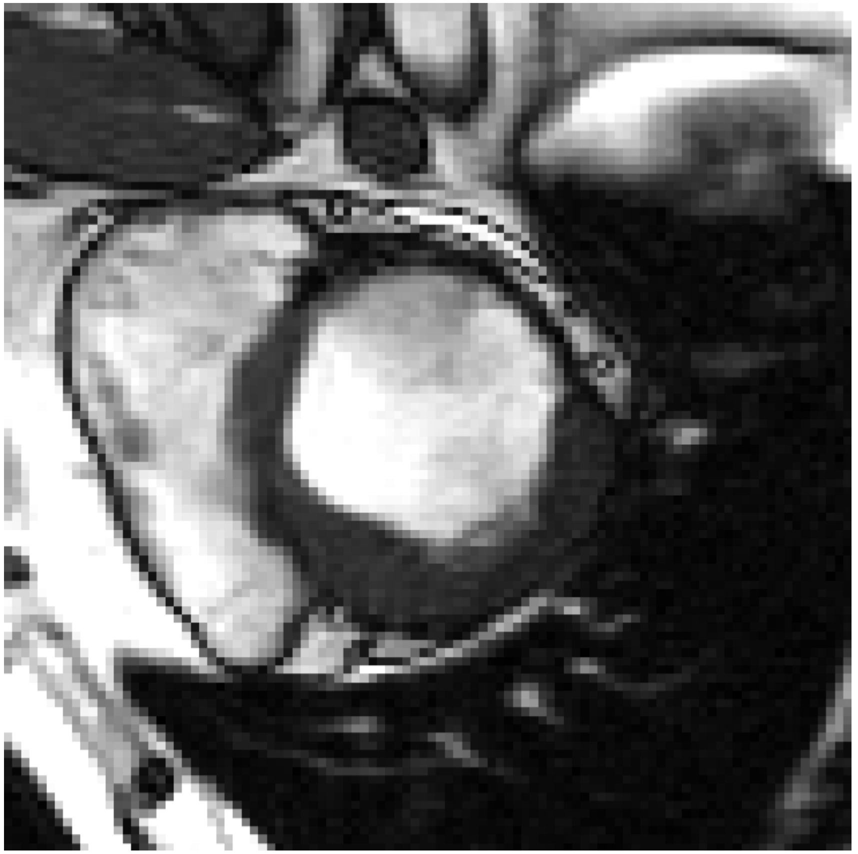
Watersheds



For topographics purposes, the watershed has been studied since the 19th century (Maxwell, Jordan, …)

Watersheds

- One hundred years later (1978), it was introduced by Digabel and Lantuéjoul for image segmentation
- And popularized by L. Vincent and P. Soille in their celebrated 1991 PAMI paper



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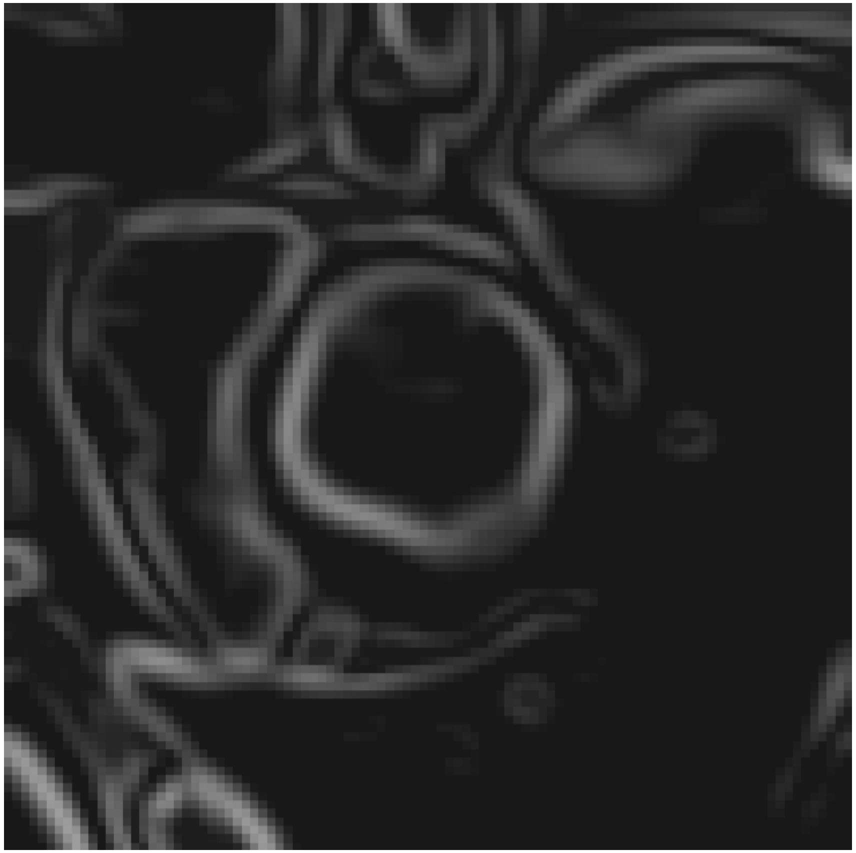

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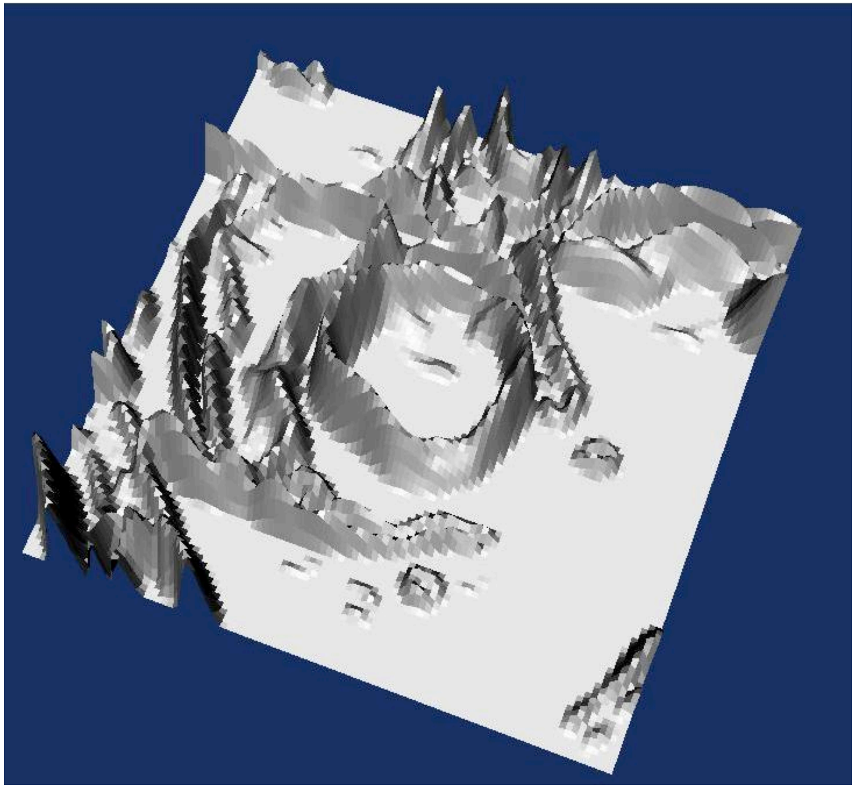

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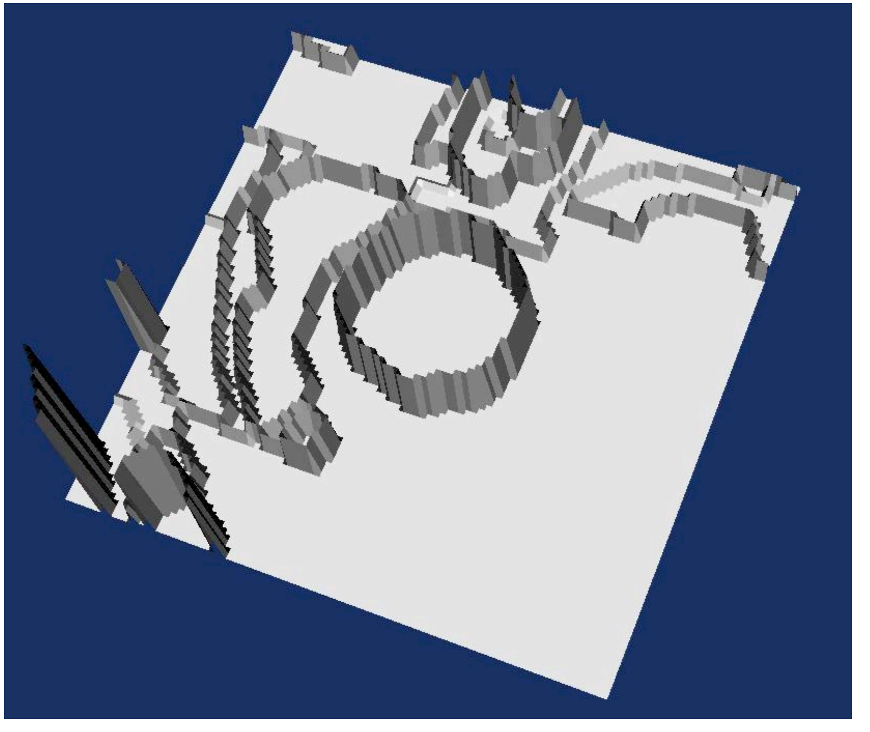

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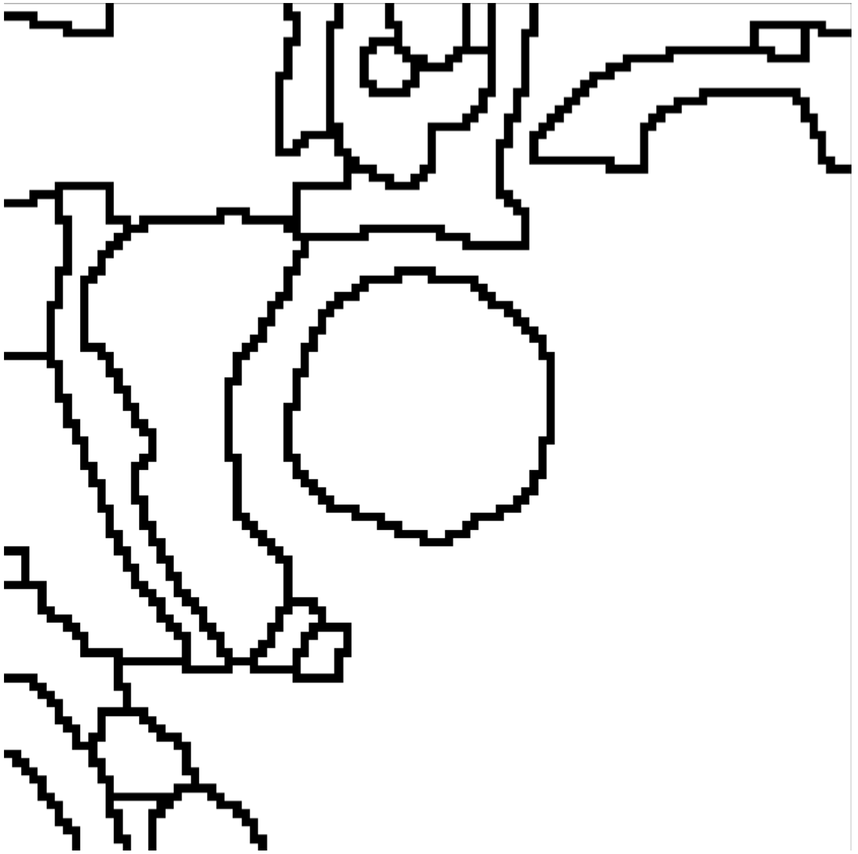

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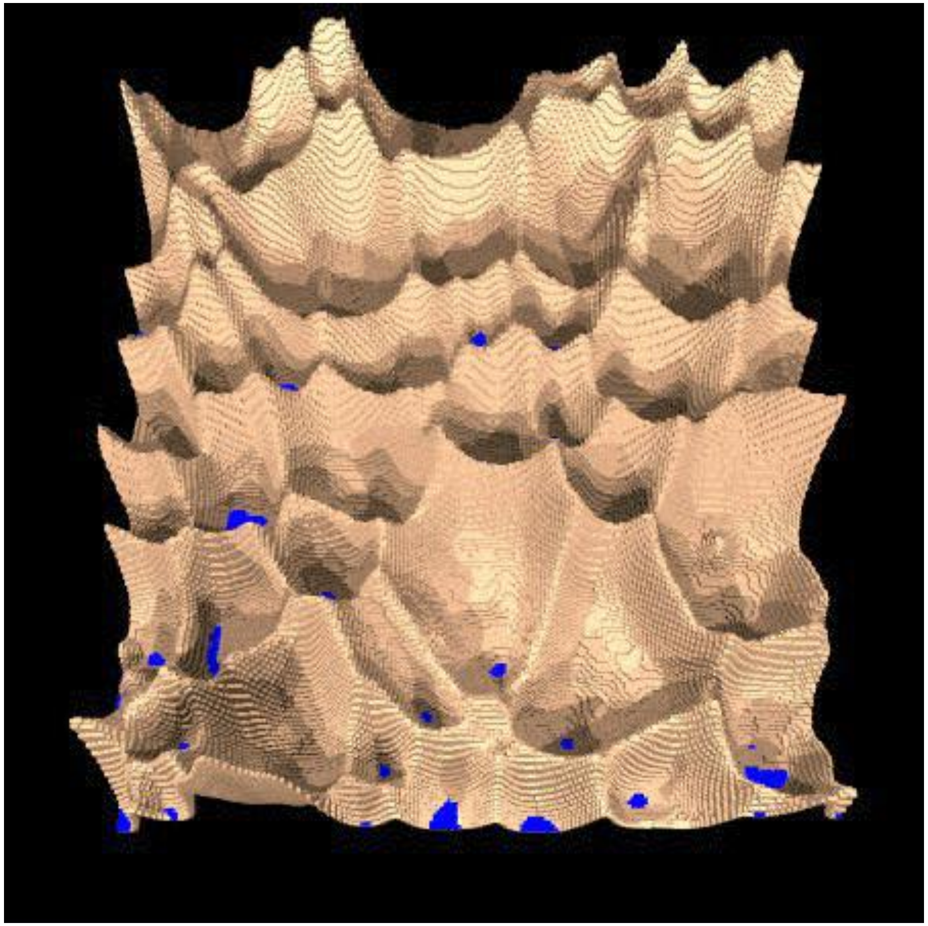

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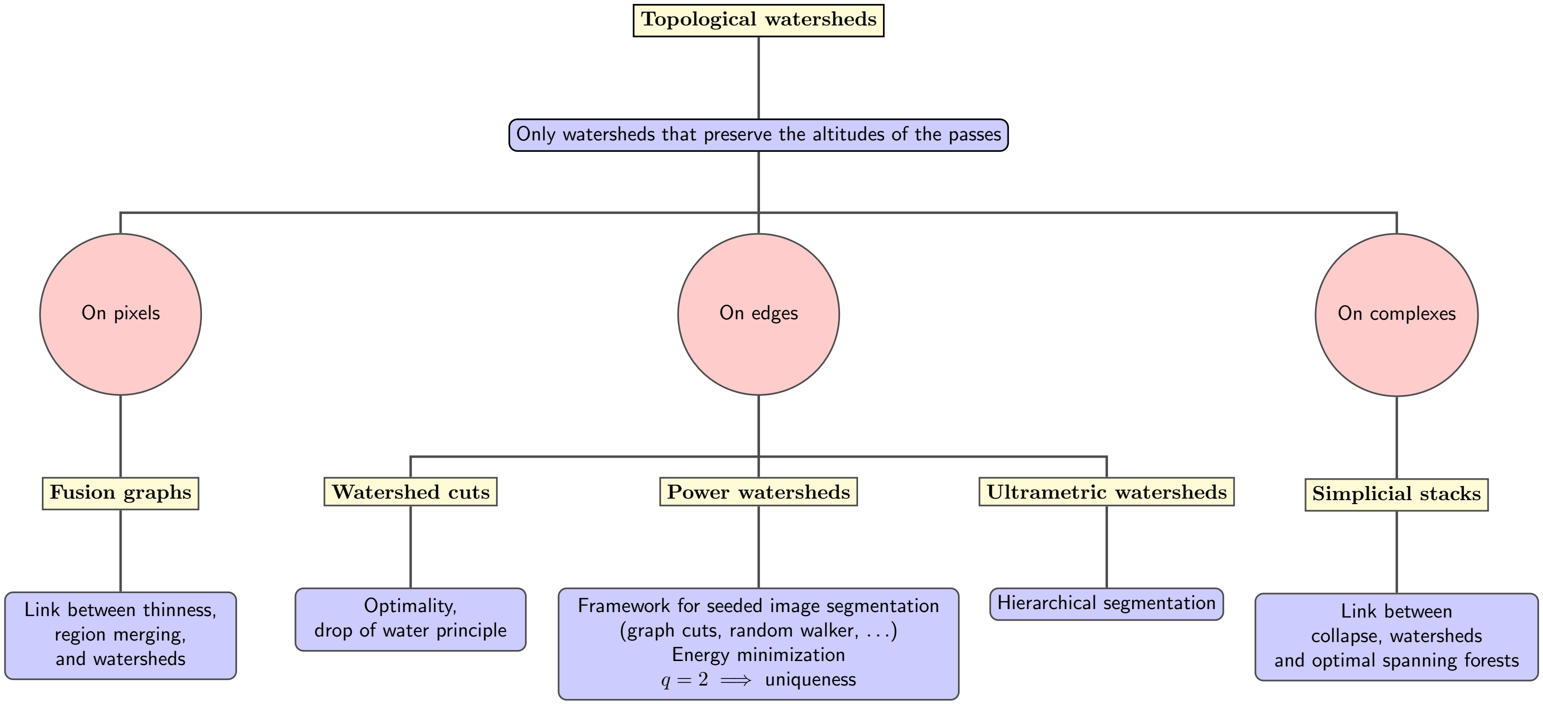
The family of discrete watersheds



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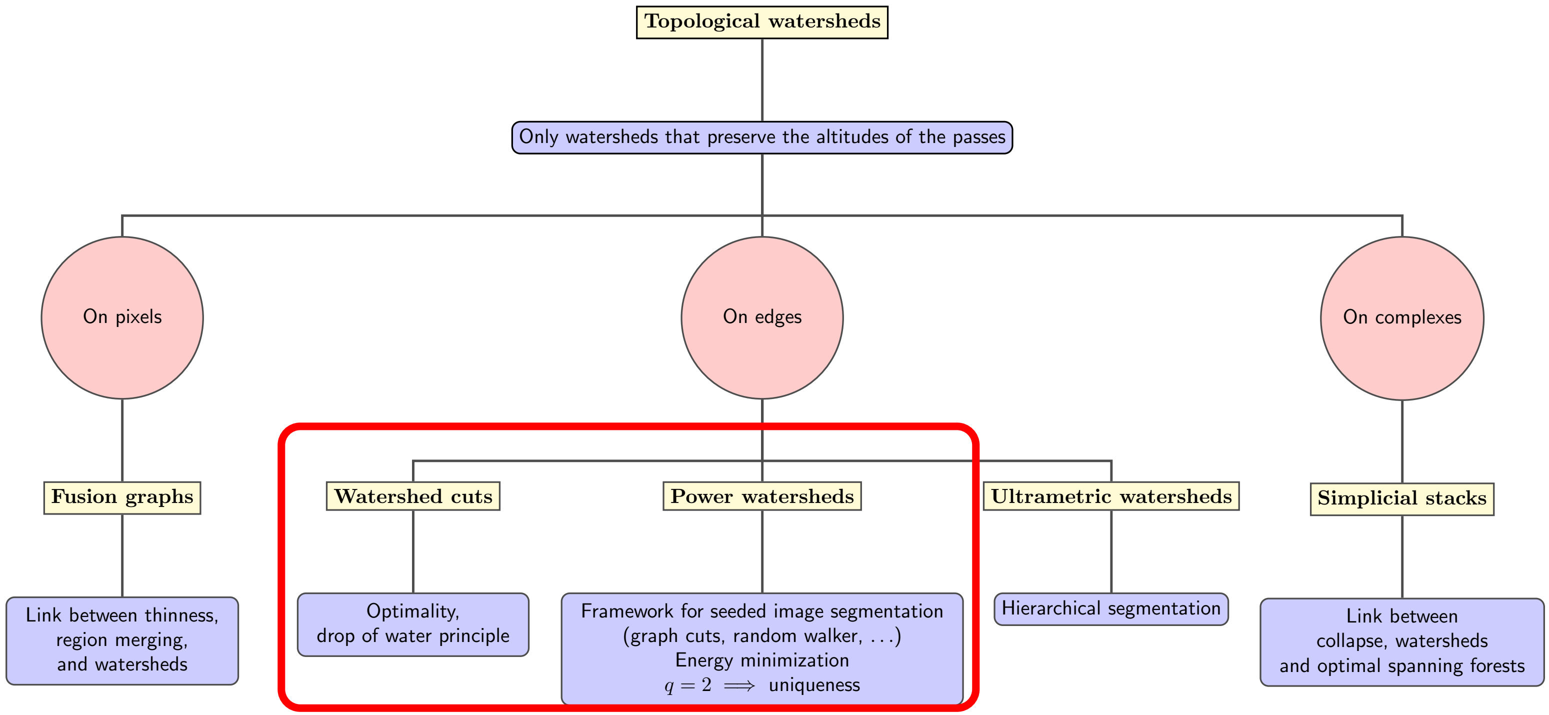
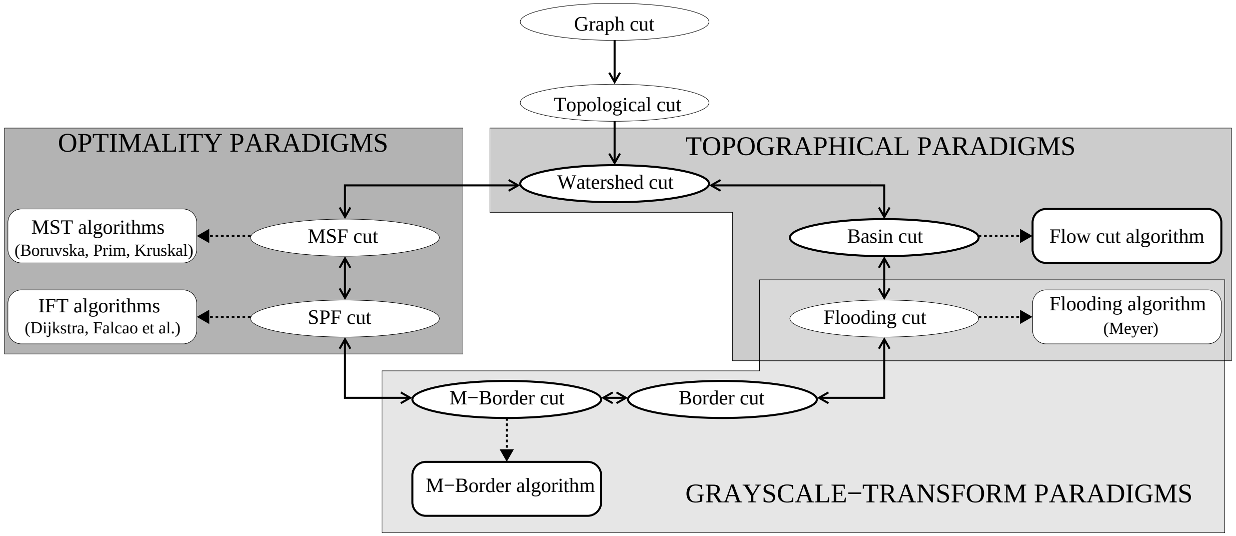


Watershed cuts

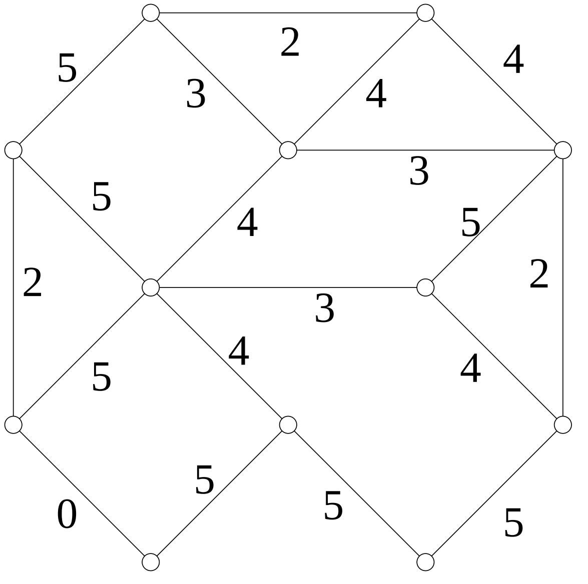
Important idea (Cousty et al., PAMI 2009, 2010)

- *Defined by the drop of water principle*
- *Equivalent to a catchment basins principle*
- *Optimal – an equivalence with Minimum Spanning Trees*



Notations

- Let $G = (V, E)$ be a graph.
- Let F be a map from E to $\mathbb{R}$.



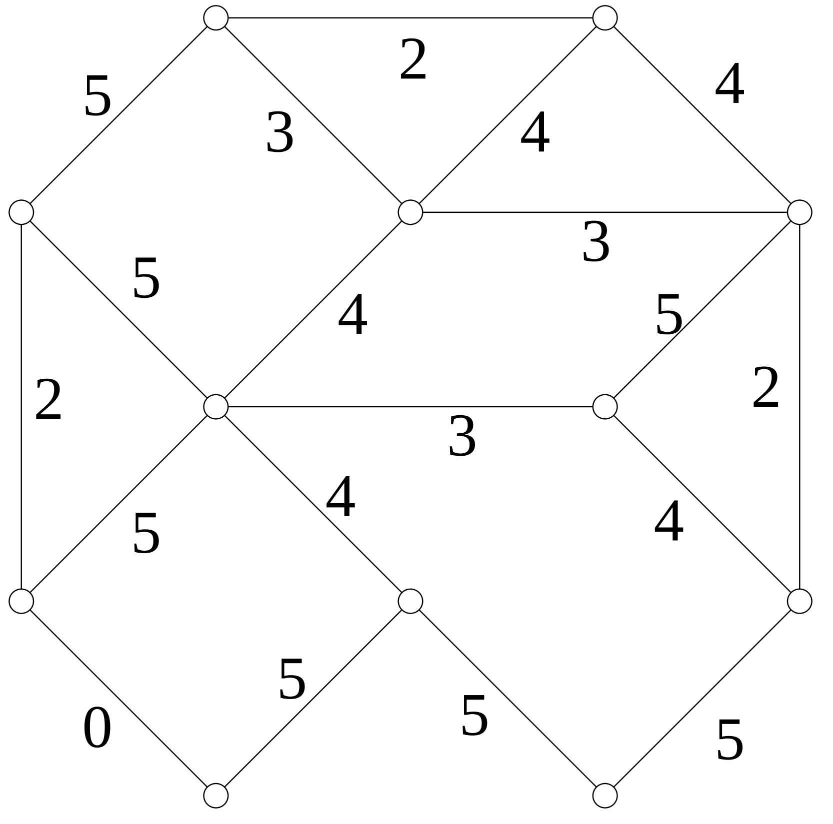
Minimum spanning forest

- The *weight of a forest* Y is the sum of its edge weights *i.e.*, $\sum_{u \in E(Y)} F(u)$.

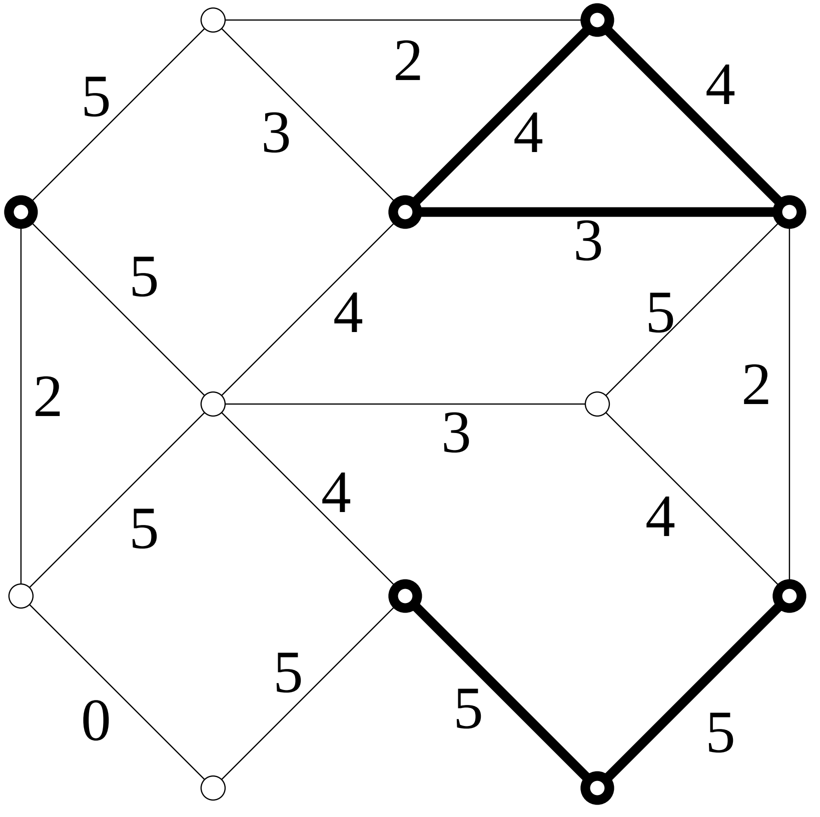
Definition

We say that Y is a minimum spanning forest (MSF) relative to X *if* Y *is a spanning forest relative to* X *and if the weight of* Y *is less than or equal to the weight of any other spanning forest relative to* X.

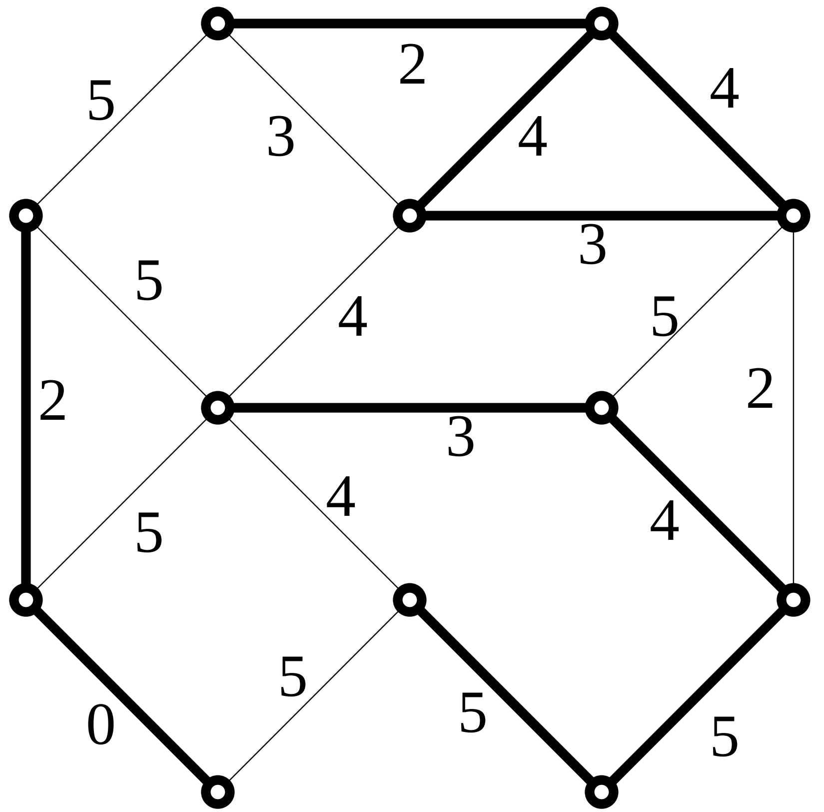
MSF - Example



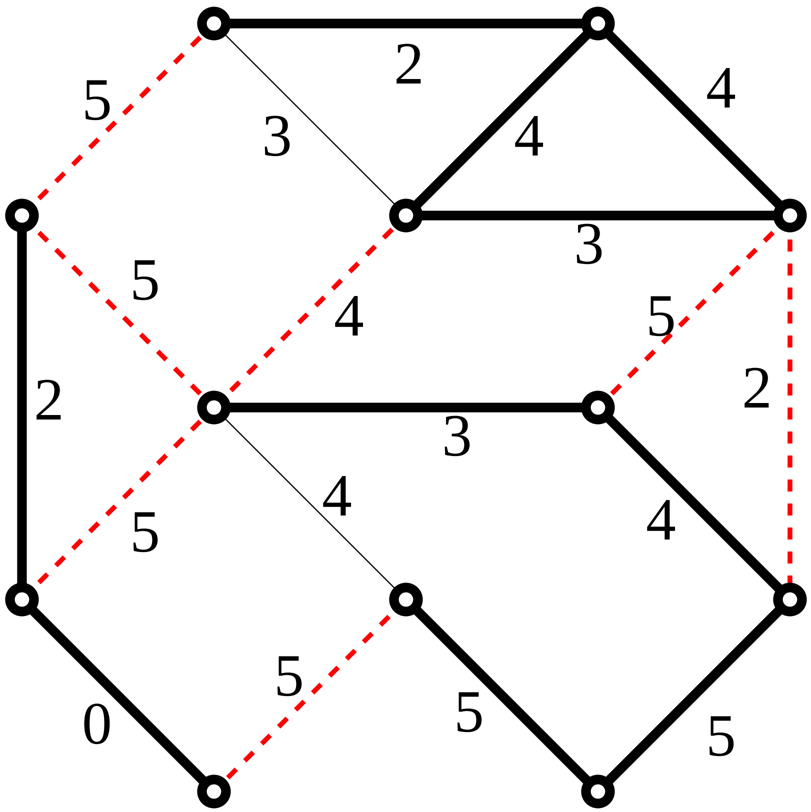
MSF - Example



MSF - Example



MSF - Example



Watershed and MSF equivalence

Theorem

An edge-set $S \subseteq E$ is a MSF cut for the minima of F if and only if *S is a watershed cut of F.*

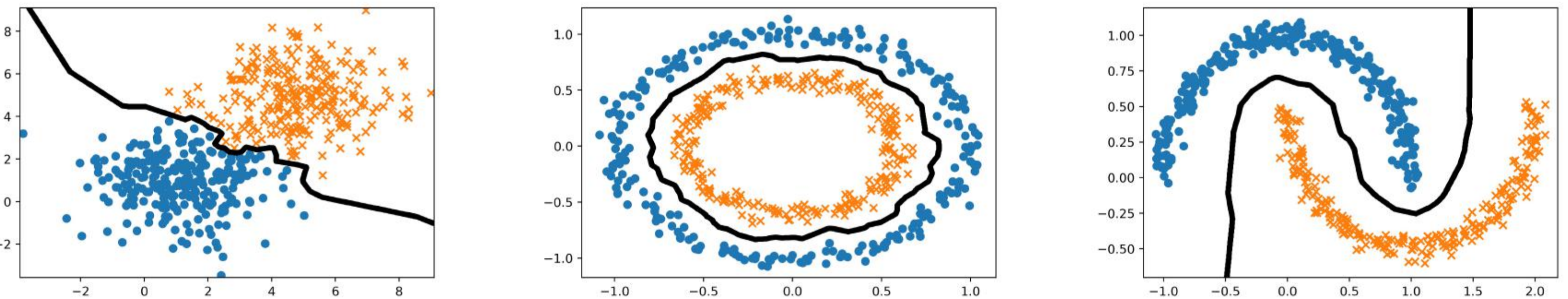
Watershed cuts as classifiers

Result

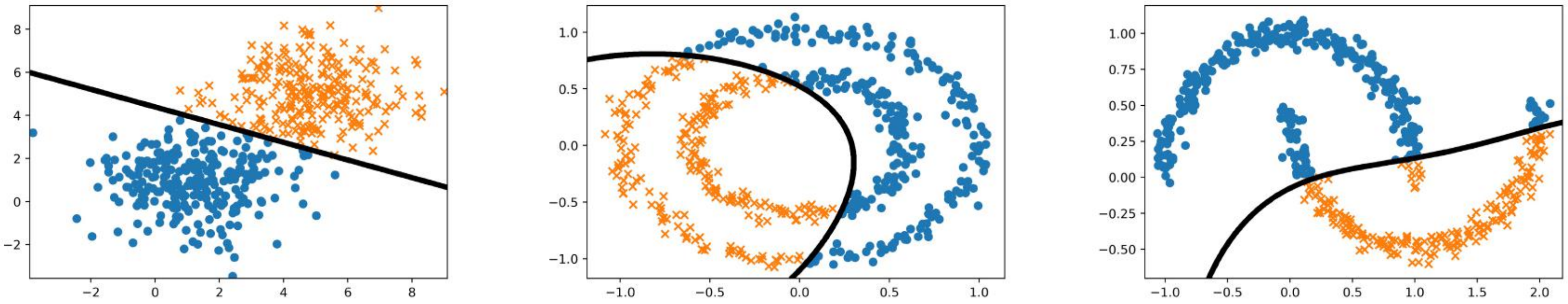
Given an edge weighted graph $G = (V, E, W)$, and a set of seeds $S = X_0 \cup X_1$, MSF-watershed returns a maximum margin partition with set of points as V and

$$\rho(x, y) = \inf_{\pi \in \Pi(x,y)} \sup_{e \in \pi} W(e)$$

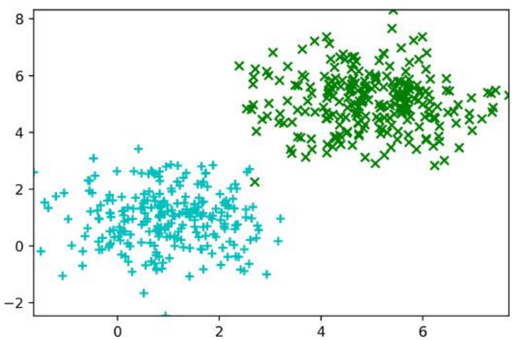
Watershed-cut as classifiers for semi-supervised learning



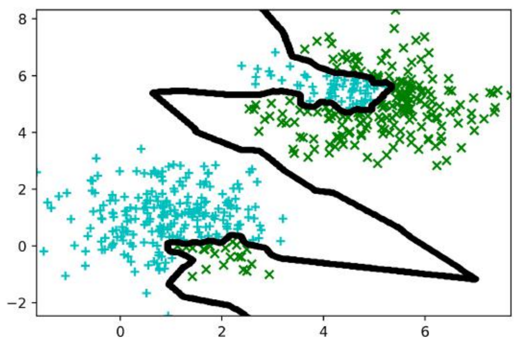
Results for SVM



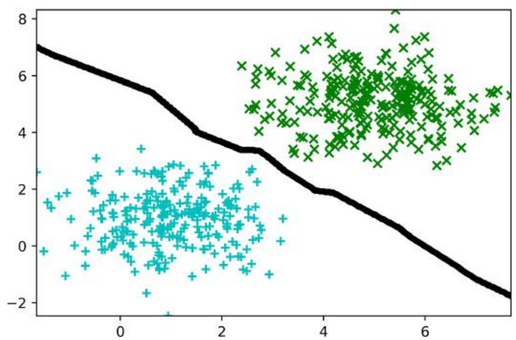
Morphological regularization



(a)



(b)



(c)

(a) Data with some wrongly labeled points
(b) MSF partition (Watershed-cut)
(c) Area-filtered watershed

Work in progress

- The watershed is a classifier
  - Hence, what if we “ensemble” watersheds?
  - Hence, what if we combine them with Neural nets

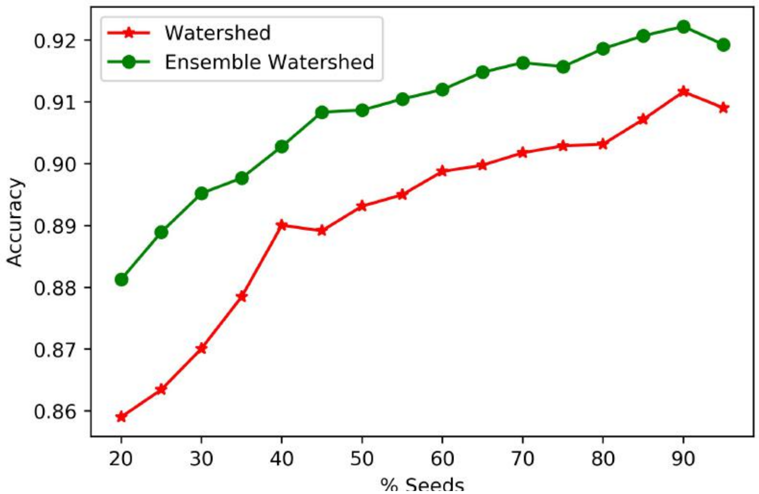
Ensemble watersheds

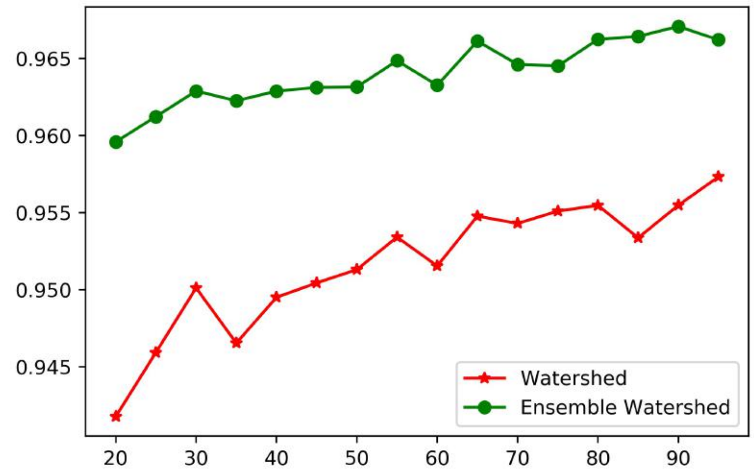
TABLE I
RESULTS OBTAINED USING DIFFERENT METHODS ON DATASETS FROM CHAPPELLE

Method	SSL1	SSL2	SSL3	SSL4	SSL5	SSL6	SSL7
watershed	96.53±0.70	95.66±0.87	99.77±0.22	51.35±3.64	55.19±1.48	95.70±0.45	54.84±1.32
IFT-SUM	96.96±0.53	95.17±0.24	95.06±1.22	53.92±2.95	61.15±0.76	90.06±0.85	64.60±1.96
RW	98.16±0.34	91.41±0.92	95.68±1.42	54.27±2.56	67.75±5.59	91.70±1.30	75.56±4.16
PW	97.99±0.49	89.42±0.78	95.68±1.42	52.22±2.29	67.75±5.59	91.68±1.36	75.56±4.16
SVM	93.78±0.67	90.81±0.57	56.87±0.81	60.00±2.81	83.57±0.80	22.16±0.49	84.49±1.22
1NN	96.96±0.53	95.18±0.24	95.06±1.22	53.92±2.95	61.15±0.75	90.06±0.85	64.61±1.97
RFC	95.36±0.87	87.74±0.58	91.42±0.64	55.76±2.33	72.75±0.89	90.51±1.06	70.45±1.75
Ensemblewatershed	98.17±0.35	92.71±1.17	99.38±0.90	53.16±3.14	64.39±3.11	95.09±0.94	68.29±1.77

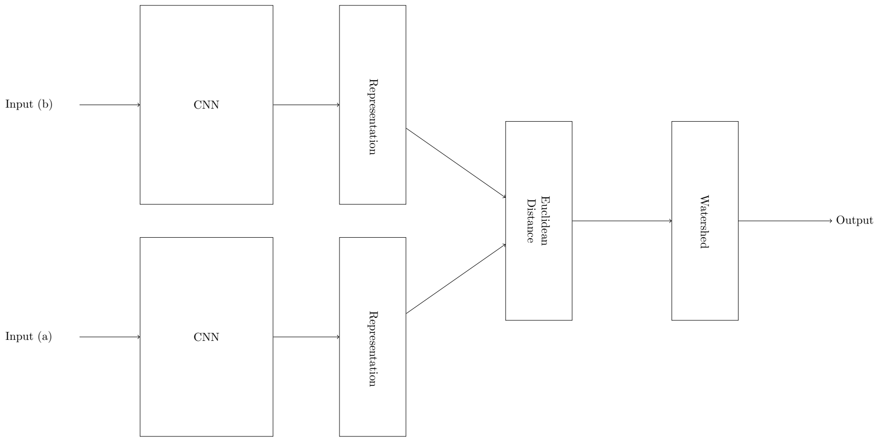
This morning results!

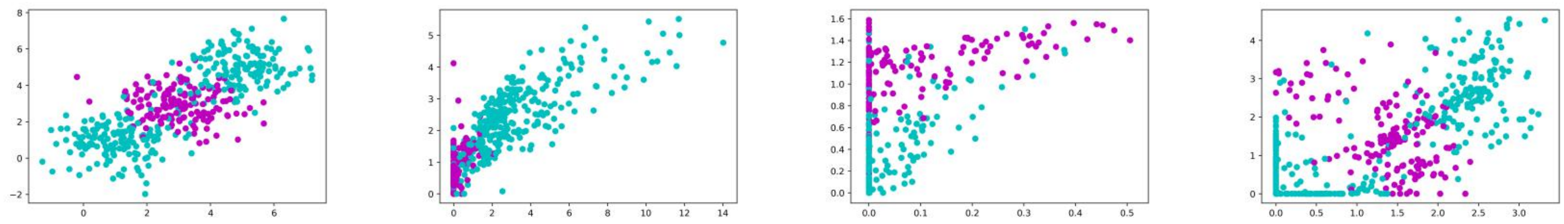
Ensemble watersheds





Using watershed as a layer of Neural Network





(a) (b) (c) (d)

(b), (c) and (d) are representation of the data by the NN

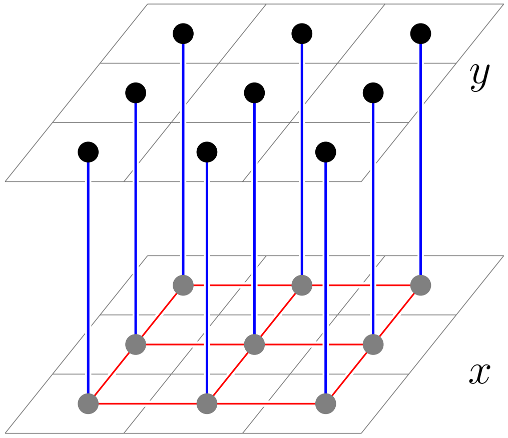
(a) Data (b) NN (c) Siamese Network (d) Siamese + WS

Preserves the structure of the data!

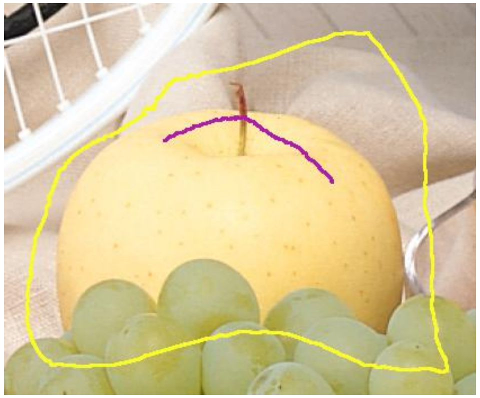
Part II: Watershed and optimization

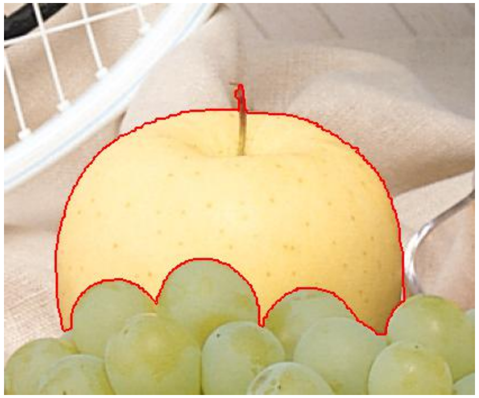
Some notations

- A simple, finite graph $G = (V, E)$ with nodes v_i and $|V| = m$
- Edge: e_{ij} spanning two vertices v_i and v_j
- Pairwise weight: w_{ij} for an edge e_{ij},
- Unary weight: w_i unary weights penalizing the (observed) configuration at node v_i.
- We are looking for x, a regularized version of the observed configuration y



A generic formulation





Power-watershed with $q \geq 0$

Let $q \geq 0$, we set

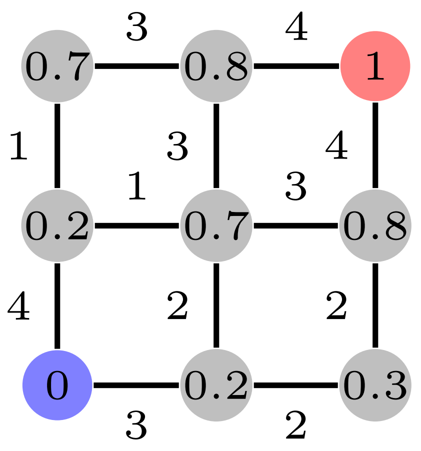
$$W^p(x) = \underbrace{\sum_{e_{ij} \in E} {w_{ij}}^p |x_i - x_j|^q}_{\text{Smoothness term}} + \underbrace{\sum_{v_i \in V} {w_i}^p |x_i - f_i|^q}_{\text{Data term}} \tag{3}$$

Random walker

- Combinatorial Dirichlet problem. [Grady 2006] (q=2)
- Resolution of system of linear equations.

Advantages

- Energy formulation $\rightarrow$ extends to a large class of problems
- No blocking artefacts



Random walker

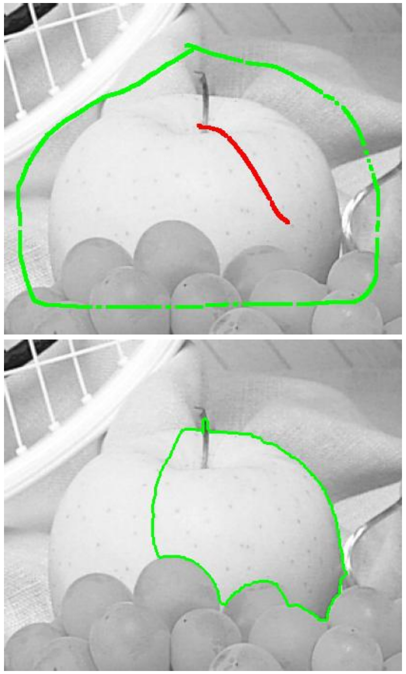
- Combinatorial Dirichlet problem. [Grady 2006] (q=2)
- Resolution of system of linear equations.

Advantages

- Energy formulation $\rightarrow$ extends to a large class of problems
- No blocking artefacts

Drawbacks

- Requires a more centered markers placement
- Super-linear complexity



And the watershed?

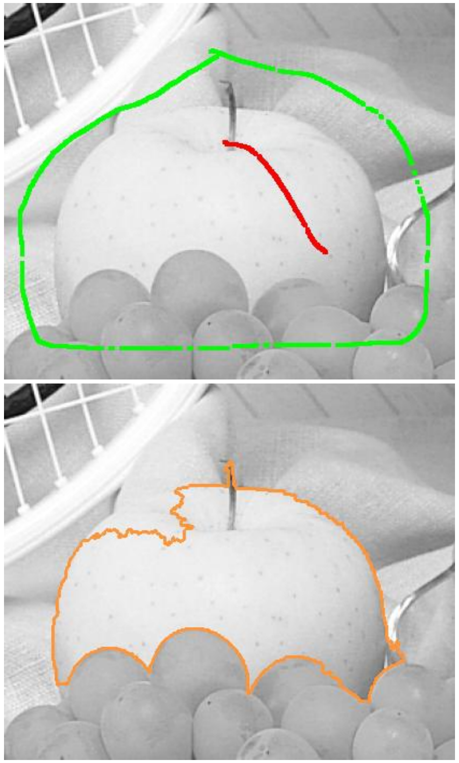
- Watershed [Beucher-Lantuéjoul 1979, Vincent-Soille 1991]

Advantages

- Fast
- Multilabel
- Robust to markers size

Drawbacks

- Leaking effect



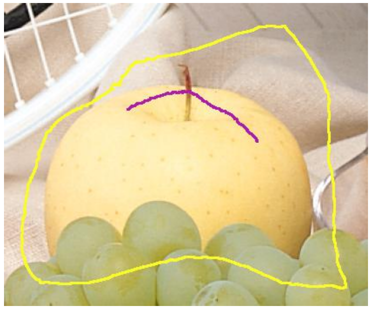
The Power Watershed framework

$$x^*_{p,q} = \arg\min_x \underbrace{\sum_{e_{ij} \in E} {w_{ij}}^p |x_i - x_j|^q}_{\text{Smoothness term}} + \underbrace{\sum_{v_i \in V} {w_i}^p |x_i - l_i|^q}_{\text{Data term}}$$

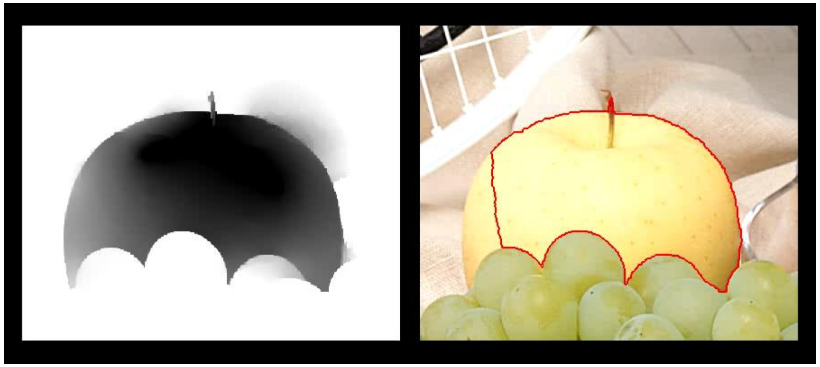
$$\bar{x} = \lim_{p \to \infty} x^*_{p,q}$$

Convergence of RW when p→∞

Input seeds



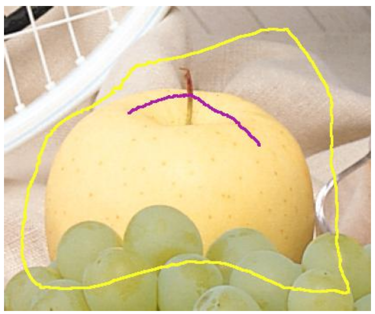
$$x^*_1 = \arg\min_x \underbrace{\sum_{e_{ij}\in E} w_{ij}^{\ 1} |x_i - x_j|^2}_{\text{Smoothness term}} + \underbrace{\mathcal{D}(x)}_{\text{Data fidelity}}$$



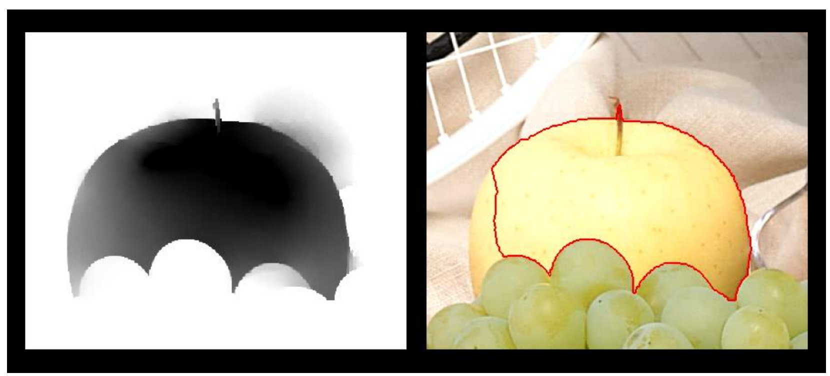
solution x^*_1 cut: threshold of x^*_1

Convergence of RW when p➔∞

Input seeds



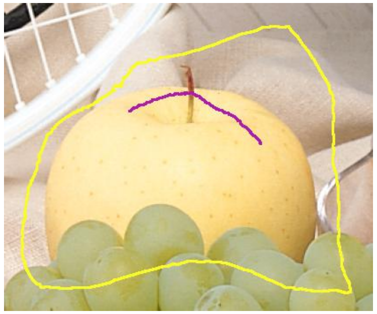
$$x^*_2 = \arg\min_x \underbrace{\sum_{e_{ij}\in E} w_{ij}^{\;2}|x_i - x_j|^2}_{\text{Smoothness term}} + \underbrace{\mathcal{D}(x)}_{\text{Data fidelity}}$$



solution x^*_2 cut: threshold of x^*_2

Convergence of RW when p→∞

Input seeds

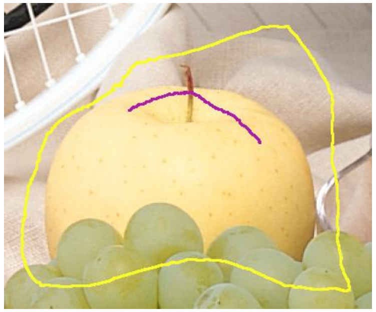


$$x^*_3 = \arg\min_x \underbrace{\sum_{e_{ij}\in E} w_{ij}^{\ 3} |x_i - x_j|^2}_{\text{Smoothness term}} + \underbrace{\mathcal{D}(x)}_{\text{Data fidelity}}$$

solution x^*_3 cut: threshold of x^*_3

Convergence of RW when p➔∞

Input seeds

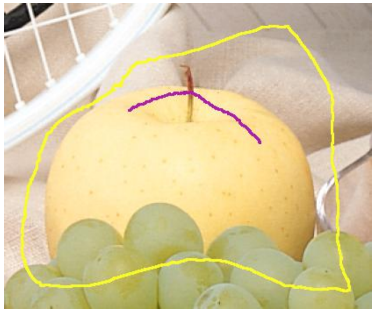


$$x^*_4 = \arg\min_x \underbrace{\sum_{e_{ij}\in E} w_{ij}^{\ 4} |x_i - x_j|^2}_{\text{Smoothness term}} + \underbrace{\mathcal{D}(x)}_{\text{Data fidelity}}$$

solution x^*_4 cut: threshold of x^*_4

Convergence of RW when p➔∞

Input seeds

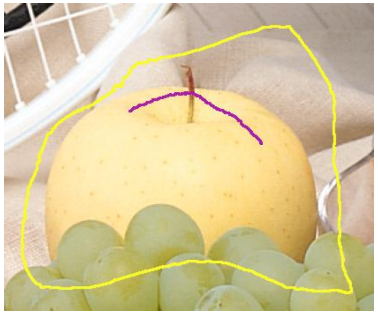


$$x^*_6 = \arg\min_x \underbrace{\sum_{e_{ij}\in E} w_{ij}^{\;6}|x_i - x_j|^2}_{\text{Smoothness term}} + \underbrace{\mathcal{D}(x)}_{\text{Data fidelity}}$$

solution x^*_6 cut: threshold of x^*_6

Convergence of RW when p→∞

Input seeds

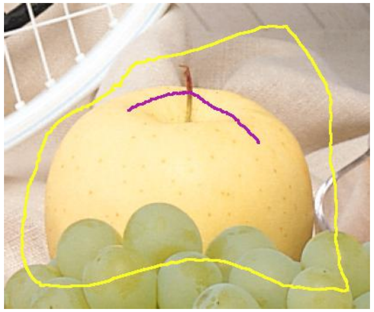


$$x^*_{9} = \arg\min_x \underbrace{\sum_{e_{ij}\in E} w_{ij}^{\ 9} |x_i - x_j|^2}_{\text{Smoothness term}} + \underbrace{\mathcal{D}(x)}_{\text{Data fidelity}}$$

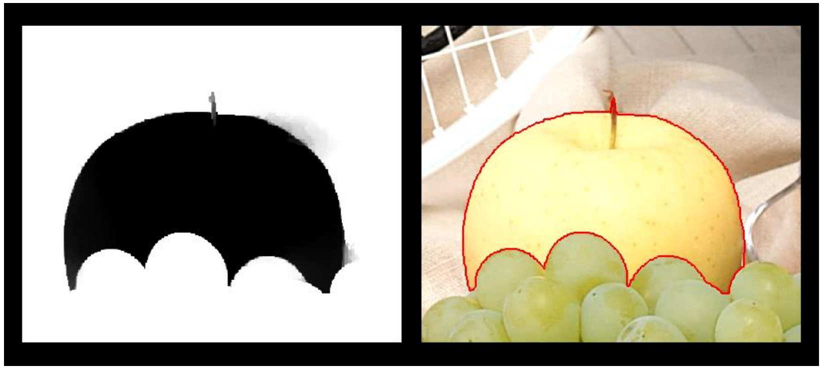
solution x^*_9 cut: threshold of x^*_9

Convergence of RW when p→∞

Input seeds



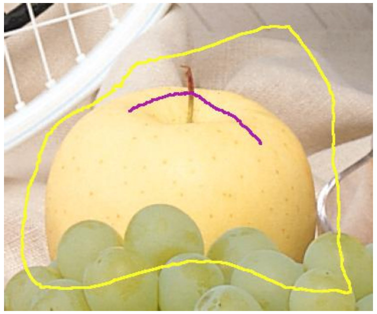
$$x_{13}^* = \arg\min_x \underbrace{\sum_{e_{ij}\in E} w_{ij}^{13}|x_i - x_j|^2}_{\text{Smoothness term}} + \underbrace{\mathcal{D}(x)}_{\text{Data fidelity}}$$



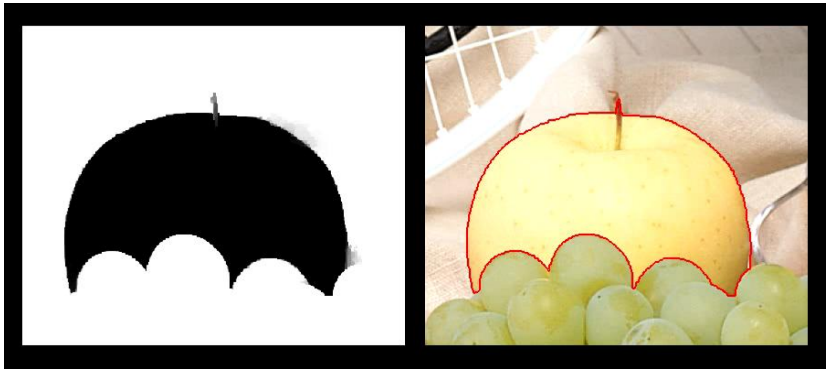
solution x_{13}^* cut: threshold of x_{13}^*

Convergence of RW when p→∞

Input seeds



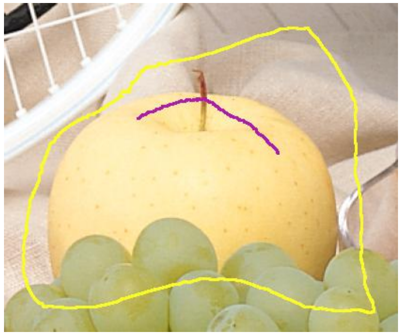
$$x_{18}^* = \arg\min_x \underbrace{\sum_{e_{ij} \in E} w_{ij}^{18} |x_i - x_j|^2}_{\text{Smoothness term}} + \underbrace{\mathcal{D}(x)}_{\text{Data fidelity}}$$



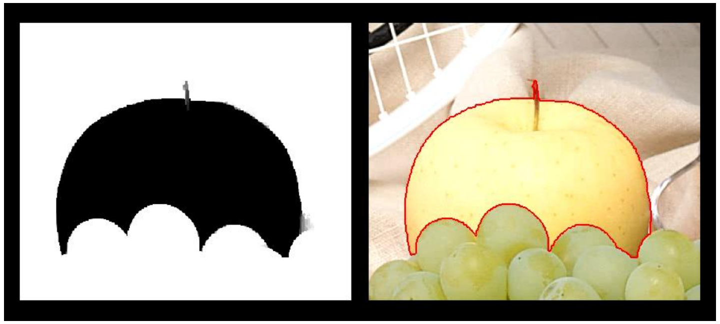
solution x_{18}^* cut: threshold of x_{18}^*

Convergence of RW when p➔∞

Input seeds



$$x_{p}^{*} = \arg\min_{x} \underbrace{\sum_{e_{ij}\in E} w_{ij}^{p} |x_i - x_j|^q}_{\text{Smoothness term}} + \underbrace{\mathcal{D}(x)}_{\text{Data fidelity}}$$



$\bar{x} = \lim_{p\to\infty} x_p^*$ cut: threshold of $\bar{x}$

Theorems

When $p \to \infty$,

- the obtained cut is an MSF cut.
- when $q > 1$, the solution $\bar{x}$ is unique.

The (extended) Power Watershed framework

- Let $p > 0$, $m > 0$, $n > 0$ and
- n real numbers $1 \geq \lambda_0 > \lambda_1 > \ldots \lambda_{n-1} > 0$

$$Q^p(x) = \sum_{0 \leq k < n} \lambda_k^p Q_k(x) \tag{1}$$

where, for all $0 \leq k < n$, $Q_k : \mathbb{R}^m \to \mathbb{R}$ is a continuous function. We search $x^\star \in \mathbb{R}^m$ such that

$$x^\star \in \lim_{p \to \infty} \operatorname*{arg\,min}_{x \in \mathbb{R}^m} Q^p(x) \tag{2}$$

Main PW theorem

Theorem

Set

$$M_0 = \arg\min_{x \in \mathbb{R}^m} Q_0(x) \tag{4}$$

$$\forall\ 1 \leq k < n,\ \ M_k = \arg\min_{x \in M_{k-1}} Q_k(x) \tag{5}$$

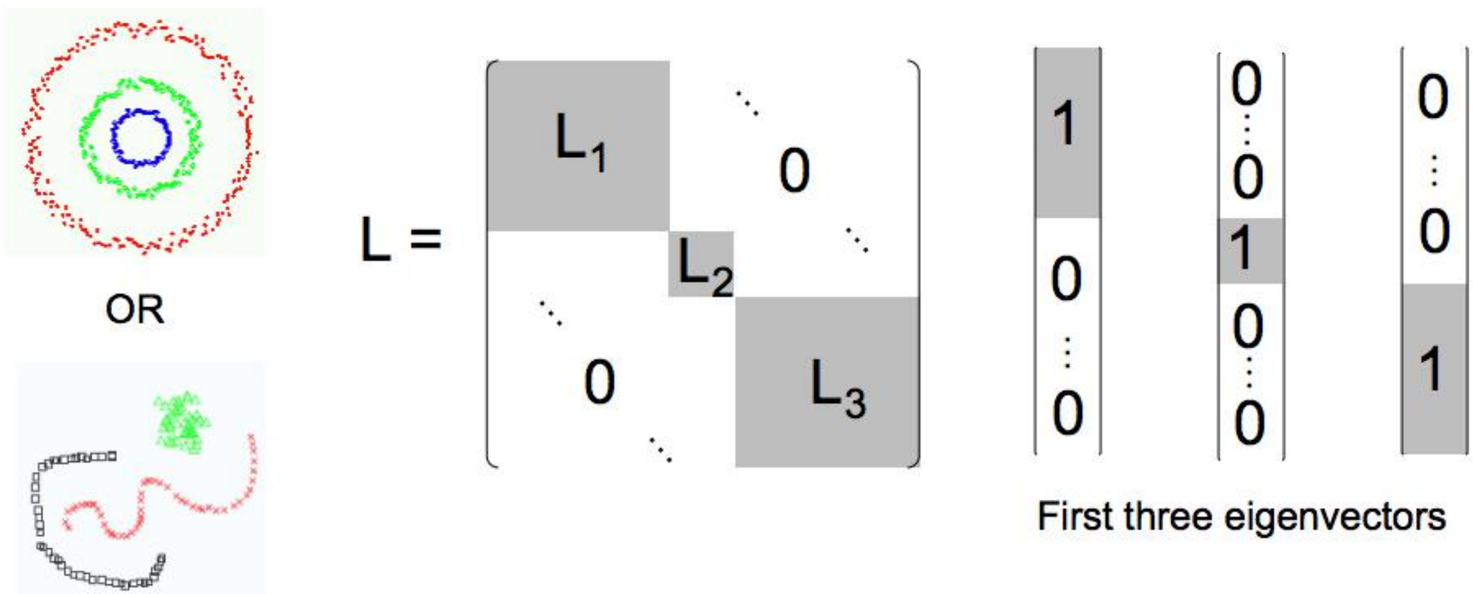
Any convergent sequence $(x_p)_{p>0}$ *of minimizers of* Q^p *converges to some point of* M_{n-1}.
Furthermore, we can estimate the minimum of Q^p *as follows:*

$$\min_{x \in \mathbb{R}^m} Q^p(x) = \sum_{0 \leq k < n} \lambda_k^p m_k + o(\lambda_{n-1}^p) \tag{6}$$

where $m_k = \min_{x \in M_k} Q_k(x)$.

Spectral clustering: intuitive explanation

Let L be one of the (many) graph-Laplacians.



Spectral clustering: ratio-cut

Problem (Ratio-cut algorithm)

For finding k cluster, solve

$$\underset{H \in \mathbb{R}^{m \times k}}{\text{minimize}} \; Tr(H^t L H)$$

$$\textit{subject to } H^t H = \mathbb{I}$$

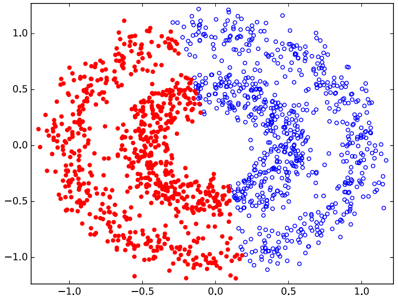
where L is the graph-laplacian

D is the diagonal matrix $diag(d_1, \ldots, d_n)$ with $d_i = \sum_j w_{ij}$, and $L = D - W$.

Ratio-cut

Let L_k as the graph-laplacian of the subgraph induced by the edges whose weights are exactly w_k.

$$\underset{H \in \mathbb{R}^{m \times k}}{\text{minimize}} \sum_{k=1}^{j} w_k \, Tr(H^t L_k H)$$
$$\text{subject to } H^t H = \mathbb{I}$$

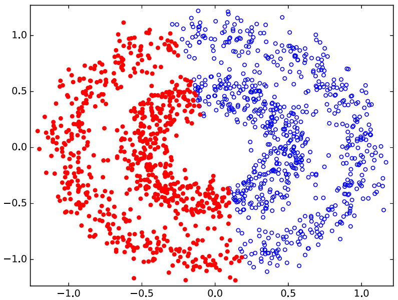


Ratio-cut

Power Ratio-cut

Let L_k as the graph-laplacian of the subgraph induced by the edges whose weights are exactly w_k.

$$\underset{H \in \mathbb{R}^{m \times k}}{\text{minimize}} \sum_{k=1}^{j} w_k^p \, Tr(H^t L_k H)$$
$$\text{subject to } H^t H = \mathbb{I}$$

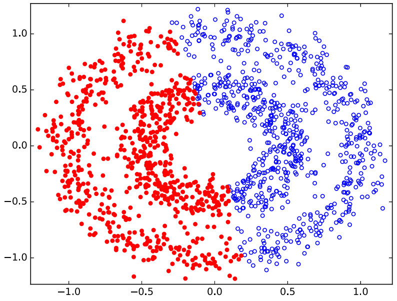


Ratio-cut

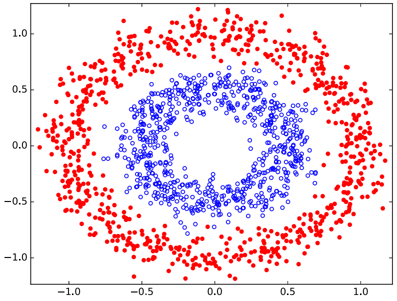
Power Ratio-cut

Let L_k as the graph-laplacian of the subgraph induced by the edges whose weights are exactly w_k.

$$\lim_{p \to \infty} \underset{H \in \mathbb{R}^{m \times k}}{\text{minimize}} \sum_{k=1}^{j} w_k^p \, Tr(H^t L_k H)$$
$$\text{subject to } H^t H = \mathbb{I}$$



Ratio-cut

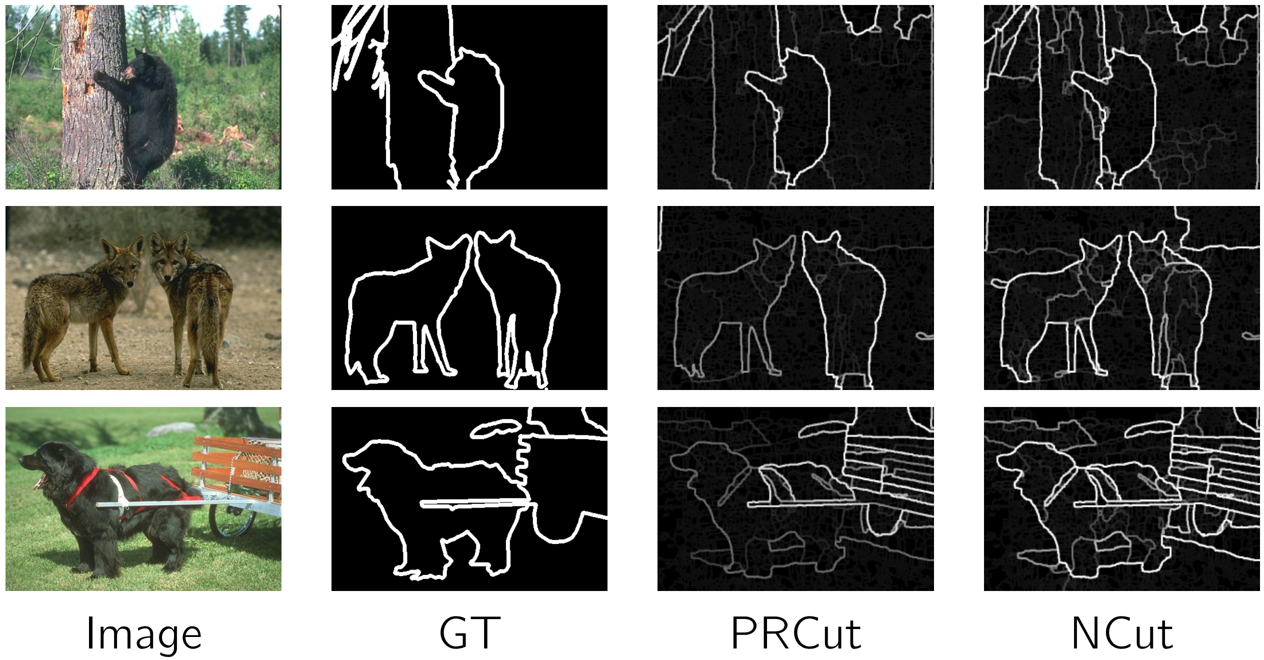


Power ratio-cut

PR-cut in practice

- Cluster the weights with (for example) K-means
- Apply a MST-like algorithm on the clustered weights to get a rough clustering
- Refine the (weighted) borders of the clusters with Ratio-cut

Replacing NCut with PRCut in MCG



Remark

Same quality of results obtained much faster
replacing Normalized Cut by Power Ratio cut
in the Multiscale Combinatorial Grouping technique

Main message

- The center of the clusters are easy to cluster
- Borders are more difficult
- Hence, apply an easy and fast algorithm on the centers (such as a MST), and do something more fancy on the borders

Question

How do we identify the borders and the centers of the cluster?

Main message

- The center of the clusters are easy to cluster
- Borders are more difficult
- Hence, apply an easy and fast algorithm on the centers (such as a MST), and do something more fancy on the borders

Question

How do we identify the borders and the centers of the cluster?

Use a MST!

MM in data science

- Is usefull ☺
- Need to revisit everything we have done
  - From a new perspective
- Much work to do!