One graph to rule them all

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Trees?

Adjacency labelling schemes

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Output: a label on k bits for each vertex of T encoding adjacencies

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Theorem (Alstrup, Dahlgaard, Knudsen '17)

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Universal graph for all graphs on $(1+o(1)) \cdot 2^{\frac{n-1}{2}}$ *vertices.*

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A key lemma for planar graphs

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood '19)

Every planar graph is the almost-induced subgraph of $P \boxtimes H$, where P is a path, and H a graph of treewidth 8.

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Theorem (Dujmović, Esperet, Joret, Walczak, Wood '19)

Planar graphs admit a nonrepetitive colouring with $O(1)$ colours.

('02 question by Alon, Grytczuk, Hałuszczak, Riordan)

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Theorem (Dujmović, Esperet, Joret, Gavoille, Micek, Morin '20)

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Lower bounds

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$\Rightarrow$ Every dense class can be optimally compressed.

Ruining the game

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Theorem (Hatami, Hatami '21)

No. (Sometimes need (almost) $2^{\sqrt{n}}$.)

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Thank you!